Kuwait University	Math 102	December 19, 1999
Dept of Math & Comp Sci	Final Examination	Duration: 2 hours

Calculators, Mobile phones and Pagers ARE NOT ALLOWED.

Answer all of the following questions. Marks for each Sub(question) are indicated in bold.

1. Let $f(x) = \tan^{-1}(\sqrt{\ln x}) + \frac{\pi}{4}$.

   (a) What is the domain of f ? Show that f is one-to-one on its domain. **(2 points)**

   (b) Show that $P(\frac{\pi}{2}, e)$ is on the graph of f^{-1} and find the equation of the tangent line to the graph of f^{-1} at P. **(2 points)**

2. Show that $\lim_{x\to 0^+} \left(\frac{5^x + 7^x}{2}\right)^{\frac{1}{x}} = \sqrt{35}$. **(2 points)**

3. Evaluate the following integrals **(4 points each)**

   (a) $\int \frac{x}{\sqrt{5+4x-x^2}}\,dx$.

   (b) $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,dx$.

   (c) $\int \frac{\sqrt{(\ln x)^2 - 1}}{x \ln x}\,dx$

   (d) $\int \frac{dx}{\sqrt{x}\,(1+\sqrt[3]{x})}$

4. Is $\int_0^{\frac{\pi}{2}} \frac{dx}{1-\sin x}$ convergent? If yes find its value. If not show why. **(4 points)**

5. Let C be a curve given parametrically by

$$x(t) = e^t \cos t, \quad y(t) = e^t \sin t, \quad 0 \le t \le 2\pi.$$

   (a) Find the points on C at which the tangent line is vertical. **(2 points)**

   (b) Find the length of C. **(2 points)**

6. Find the area of the region that is outside the graph of the polar equation $r = 3(1+\cos\theta)$ and inside the graph of the polar equation $r = 4 + \cos\theta$. **(4 points)**

7. Given the points $P(1,1,1)$, $Q(3,-1,2)$ and $R(1,0,0)$

   (a) Find the area of the triangle determined by P, Q, and R. **(2 points)**

   (b) Find a parametric equations of the line through P and Q. **(2 points)**

   (c) Find the equation of the plane containing P, Q, and R. **(2 points)**

1. (a) $D_f = [1,\infty)$

$\frac{df}{dx}(x) = \frac{1}{2x\,(1+\ln x)\,\sqrt{\ln x}} > 0, \quad x > 1 \Rightarrow f$ is increasing on $[1,\infty)$. Thus, f is $(1-1)$.

(b) $f(e) = \tan^{-1}(1) + \frac{\pi}{4} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow P$ on the graph of f^{-1}.

$f'(f^{-1}(\frac{\pi}{2})) = f'(e) = \frac{1}{4e}$. The slope of the tangent line $= 4e$.

The equation of the tangent line is $y = 4e(x - \frac{\pi}{2}) + e = 4ex + (1-2\pi)e$

2. $\lim_{x\to 0^+}\left(\frac{5^x+7^x}{2}\right)^{\frac{1}{x}} = e^{\lim_{x\to0^+}\frac{\ln(5^x+7^x)-\ln 2}{x}} = e^{\lim_{x\to0^+}\left(\frac{5^x\ln 5 + 7^x \ln 7}{5^x+7^x}\right)} = e^{\frac{1}{2}\ln 35} = \sqrt{35}$

3. (a) $5+4x-x^2 = 9-(x-2)^2$. $\quad u = x-2, \quad du = dx$

$$\int \frac{x}{\sqrt{5+4x-x^2}}\,dx = \int \frac{u+2}{\sqrt{9-u^2}}\,du = \int \frac{u}{\sqrt{9-u^2}}\,du + \int \frac{2}{\sqrt{9-u^2}}\,du$$
$$= -\sqrt{9-u^2} + 2\sin^{-1}\tfrac{u}{3} + C = -\sqrt{5+4x-x^2} + 2\sin^{-1}\tfrac{x-2}{3} + C.$$

(b) $$\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,dx = 2\int (\sqrt{x})'\tan^{-1}\sqrt{x}\,dx = 2\left[\sqrt{x}\tan^{-1}\sqrt{x} - \tfrac{1}{2}\int \tfrac{\sqrt{x}}{1+x}\left(\tfrac{1}{\sqrt{x}}\right)dx\right]$$
$$= 2\sqrt{x}\tan^{-1}\sqrt{x} - \int_1^\infty \tfrac{1}{1+x}dx = 2\sqrt{x}\tan^{-1}\sqrt{x} - \ln|x+1| + C$$

(c) $u = \ln x, \ du = \frac{1}{x}dx$

$$\int \frac{\sqrt{(\ln x)^2-1}}{x\ln x}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du \overset{u=\sec\theta}{=} \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$
$$= \tan\theta - \theta + C = \sqrt{u^2-1} - \sec^{-1}u + C = \sqrt{(\ln x)^2-1} - \sec^{-1}(\ln x) + C.$$

(d) $u^6 = x, \quad 6u^5du = dx$

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt[3]{x})} = \int \frac{6u^5}{u^3\,(1+u^2)}du = 6\int \frac{u^2}{1+u^2}du = 6\int du - 6\int \frac{1}{1+u^2}du$$
$$= 6u - 6\tan^{-1}u + C = 6\sqrt[6]{x} - 6\tan^{-1}\sqrt[6]{x} + C.$$

4. $\int_0^t \frac{dx}{1-\sin x} = \int_0^t \frac{1+\sin x}{\cos^2 x}dx = \int_0^t \left[\sec^2 x + (\cos x)^{-2}\sin x\right]dx = \tan t + \frac{1}{\cos t} \to \infty$, as $t \to \frac{\pi}{2}^+$. The integral diverges.

5. (a) m =The slope of the tangent line at $(x(t),y(t)) = \frac{y'(t)}{x'(t)} = \frac{\sin t + \cos t}{\cos t - \sin t}$.

The tangent is vertical $\Leftrightarrow x'(t) = \cos t - \sin t = 0 \Leftrightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}$.

Points are $(x(\frac{\pi}{4}), y(\frac{\pi}{4})) = (\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}, \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}})$, and $(x(\frac{5\pi}{4}), y(\frac{5\pi}{4})) = (-\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}, -\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}})$.

(b) The length of $C = \int_0^{2\pi} \sqrt{2e^{2t}}\,dt = \sqrt{2}\,[e^t]_{t=0}^{t=2\pi} = \sqrt{2}(e^{2\pi}-1)$.